GRAVITATIONAL LENSING AND A FIRST TEST OF EXOTIC TOPOLOGY

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Fig. 1: One of the strongest pieces of evidence in favor of dark matter is the Bullet Cluster, where most of the visible mass is associated with hot, x-ray emitting gas (pink), but the vast majority of total mass is located in a different location (blue). Critical to this argument is the reconstruction of total mass distribution using *gravitational lensing*.

Dark Matter—New Cosmological Epicycle?

Dark matter is hypothesized to make up 25% of the energy content of the Universe [1]. It cannot consist of anything within the existing Standard Model of particle physics, and has not been directly detected in any experiment. The evidence for dark matter is circumstantial: it seems to be needed in order to explain the behavior of visible matter, from the rotational velocities of stars in spiral galaxies to the growth and distribution of large-scale structures like clusters of galaxies in the early universe. Perhaps the strongest evidence for dark matter comes from gravitational lensing (Fig. 1), which allows us to reconstruct the location of gravitating mass even where none is seen.

But the fact that we need to invoke an unseen entity to reconcile modern cosmology with observation strikes some as reminiscent of the epicycles of Ptolemaic astronomy. Einstein's General Relativity (GR) teaches us that "gravity = curved spacetime." Could it be that what we have taken as the gravitational effects of new forms of matter-energy is really just a manifestation of *exotic spacetime structure*—i.e., topologically more complex than $\mathbb{R}^4$?

Exotic Topology

Topology studies properties of space and geometry that are preserved under continuous deformations (homeomorphisms and diffeomorphisms). Homeomorphisms are invertible transformations that do not involve cutting or gluing. Diffeomorphisms are differentiable homeomorphisms, where we can perform calculus.

The property of interest here is "exotic smoothness." Exotic manifolds are manifolds that are homeomorphic, but not diffeomorphic, to n-dimensional Euclidean space $\mathbb{R}^n$. Exotically smooth manifolds, in other words, are those in

Fig. 2: Artist's interpretation of exotic four-dimensional space [2]. The center is the Euclidean representation of four-space, while the tendrils represent its exotic nature as it approaches infinity.

which we can do calculus ("smooth"), but which do not map smoothly to ordinary space ("exotic"). Such a space is depicted schematically in Fig. 2.

The first exotic versions of ordinary four-dimensional space $\mathbb{R}^4$ were discovered by Michael Freedman in 1982 and Simon Donaldson in 1983 [3]. Their work was extended by Robert Gompf in 1985 and Clifford Taubes in 1987, who showed that there are, in fact, an *uncountable infinity* of exotic versions of four-dimensional Euclidean space [4]. Most remarkably, these exotic versions of ordinary space *exist only in the case of four dimensions*.

These results may have earthshaking physical significance, yet they have hardly been noticed by physicists so far. Is it a coincidence that real spacetime is four-dimensional? A basic requirement for any practical field theory is that the underlying space be differentiable ("smooth"). But need it be Euclidean? According to GR, what we feel as the "force of gravity" is actually just a manifestation of curved spacetime. But if that spacetime is Euclidean, observations imply that it is curved by vast amounts of unseen dark matter and energy. Could it be, instead, that spacetime is exotic? This idea is known as the "Brans conjecture" [5].

Gravitational Lensing Test

The challenge is to extract, from the uncountable infinity of possible exotic $\mathbb{R}^4$s, a space whose properties can be described by an actual spacetime metric. A major first step in this direction has been taken by Chris Duston, who decomposed exotic spacetime using topological objects known as "Casson handles" and recombined them to obtain a metric using a mathematical generalization of the Fourier transform known as the Z-transform [6]. He has applied this method to obtain two different metrics, one astrophysical (a Kruskal-like "exotic black hole") and the other cosmological ("exotic Friedmann-Robertson-Walker" or eFRW). Due to the details of the Z-transform, the cosmological metric bifurcates into two possible cases, one describing 4D spacetime and the other 3D space only:

$$ds^2 = \begin{cases} -dt^2 + a(t)^2\left(\tfrac{1}{2}dr^2 + r^2 d\Omega^2\right) & 0 \le t < 1 \\ a(t)^2\left(\tfrac{1}{2}\right)k^{n/2}dr^2 + r^2 d\Omega^2 & t \ge 1 \end{cases} . \quad (1)$$

Here $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2$ and $a(t)$ is the usual cosmological scale factor, but k is not the curvature parameter (this model is flat). We restrict our attention here to the 4D case ($0 \le t < 1$) and follow Duston in assuming that $\dot{a}(t) \approx 0$ for the purposes of our lensing calculation.

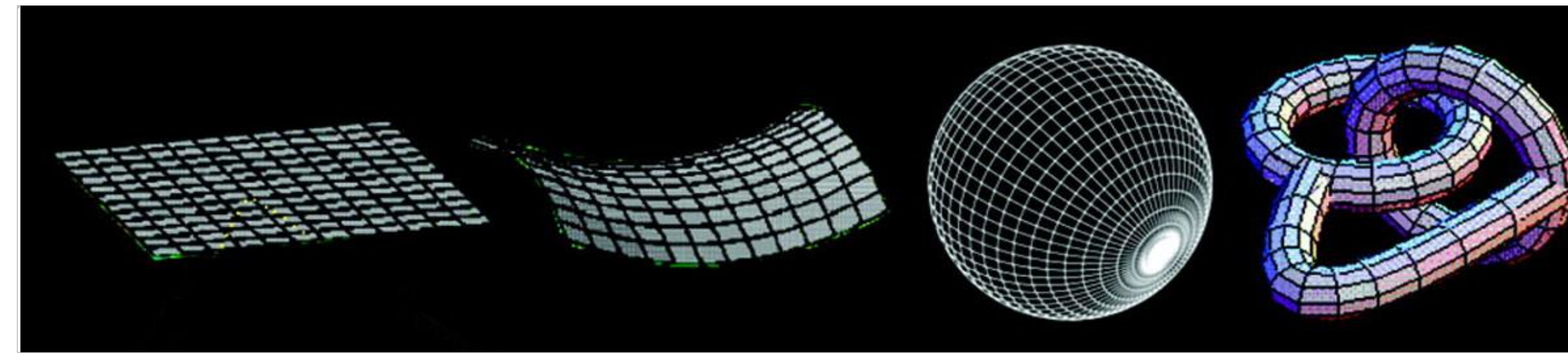

Fig. 4: Using light deflection to probe the global topology of space (taken from [8]). The three figures at left show the three possibilities for standard (Euclidean) topology: flat, open and closed. The figure at right is meant to suggest a possible topological alternative (not to scale!)

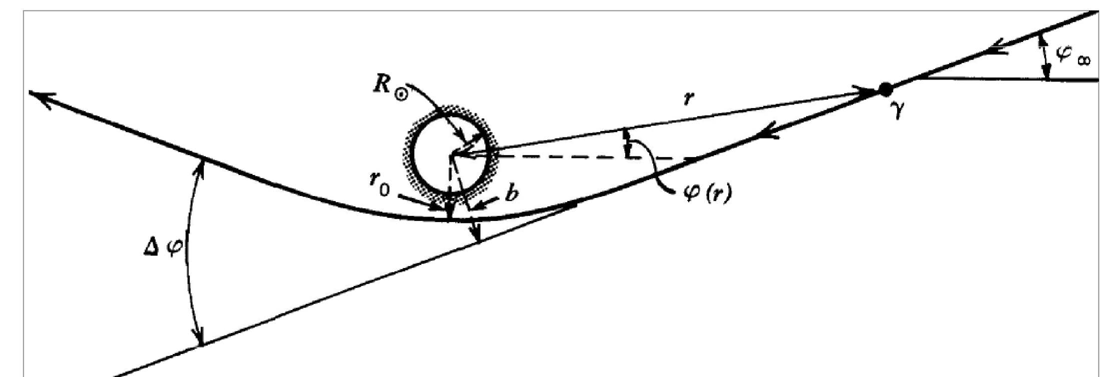

Fig. 3: Setup and important quantities for light deflection calculation [7]

With this approximation, Eq. (1) takes the static, spherically symmetric form studied by Weinberg in his analysis of light deflection in standard GR [7]:

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega^2 \ . \quad (2)$$

From the dt^2 term we see that $B(r) = 1$; and from the dr^2 and $d\Omega^2$ terms we see that $A(r) = \frac{1}{2}$ if we rescale the radial coordinate $r \to ar$. To obtain the lensing angle, we solve the geodesic equation. The t, r components of this equation give two conserved quantities (energy per kilogram E and angular momentum per kilogram J), which we use to solve the φ equation (the θ equation is satisfied automatically by symmetry). This results in a differential equation for $r(\varphi)$:

$$\frac{A(r)}{r^4}\left(\frac{dr}{d\varphi}\right)^2 + \frac{1}{r^2} - \frac{1}{J^2 B(r)} = -\frac{E}{J^2} \ , \quad (3)$$

where $J = bV^2, E = 1 - V^2$, and b is the impact parameter of a test body approaching the lensing mass from infinitely far away at velocity V. Total deflection $\Delta\varphi = 2|\varphi(r_0) - \varphi_\infty| - \pi$, where r_0 is the distance of closest approach to the lensing mass and φ_∞ is the value of φ at infinity (Fig. 3). Solving Eq. (3) with $V^2 = 1$ for photons, we obtain

$$\varphi(r) - \varphi_\infty = \int_r^\infty A^{1/2}(r)\left[\left(\frac{r}{r_0}\right)^2\left(\frac{B(r_0)}{B(r)}\right) - 1\right]^{-1/2}\frac{dr}{r}$$

In GR with a Schwarzschild metric, $A(r) = 1 + \frac{2GM}{c^2 r}$ and $B(r) = \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \approx 1 - \frac{2GM}{c^2 r}$, and we find that

$$\varphi(r_0) - \varphi_\infty = \frac{\pi}{2} + \frac{2GM}{c^2 r_0} \quad \text{so} \quad \Delta\varphi = \frac{4GM}{c^2 r_0} . \quad (4)$$

By contrast, with Duston's expressions for $A(r)$ and $B(r)$ as above, we find instead

$$\varphi(r_0) - \varphi_\infty = \frac{\pi a}{2\sqrt{2}} \quad \text{so} \quad \Delta\varphi = \pi\left(\frac{a}{\sqrt{2}} - 1\right) . \quad (5)$$

For smooth exotic topology to mimic the effects of lensing by a mass M in standard GR, Eqs. (4) and (5) imply that the "topological parameter" a must be given by

$$a = \sqrt{2}\left(1 + \frac{4GM}{\pi c^2 r_0}\right) \ . \quad (6)$$

In the context of light deflection in the solar system, this identification would appear rather fine-tuned, since the second term in parentheses in Eq. (6) is only $\sim 3 \times 10^{-6}$.

But for cosmological lensing, as seen in the angular size of fluctuations in the cosmic microwave background, the situation is different. Here, as a rough guess, we take $r_0 \sim ct_0$ where t_0 is the age of the universe, $M = \rho_{\text{crit}} V$ where $\rho_{\text{crit}} = 3H_0^2/8\pi G$, the Hubble expansion rate $H_0 = 1/t_0$, and $V \sim \frac{4}{3}\pi r_0^3$. Combining these expressions, we find that the second term in the parentheses in Eq. (6) is now $2/\pi$; i.e., of order unity. The implication is that exotic smoothness might indeed explain gravitational lensing on cosmological scales in a natural way.

Discussion

The assumption that $a(t) =$ constant in the metric (1) is a severe limitation that should be lifted in further work. Nonetheless, *we have here obtained the first approximate empirical constraint on smooth exotic topology as an alternative to dark matter*.

Acknowledgments

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